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Stronger violation of local theories with equalities

Hossein Movahhedian

Department of Physics, Shahrood University of Technology, Seventh Tir Square, Shahrood, Iran

E-mail: hossein_movahhedian@catholic.org

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Abstract
Bell-type inequalities are used to test local realism against quantum theory. In this paper, we consider a two-party system with two settings and two possible outcomes on each side, and derive *equalities* in local theories which are violated by quantum theory by a factor of 1.522 tolerating 0.586 fraction of white noise admixture which is twice that of the previous results.

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1. Introduction

The idea of local realism opened its way into quantum theory by the work of Einstein, Podolsky and Rosen which is well known as EPR paradox [1], though it is not really a paradox, because it has been argued recently that the assumptions of EPR were wrong [2]. However, it was left to John S Bell who derived an inequality based on local theories and proved that it was violated by statistical predictions of quantum theory [3, 4]. Since then many attempts have been made to derive Bell-type inequalities which are violated by a stronger factor, so that it can be tested in the real experiments in which errors are inevitable. Among these are Clauser, Horne, Shimony and Holt inequality, the so-called CHSH inequality [5, 6].

Nowadays, with the growing power of computers, the numerical methods have attracted attentions for constructing these inequalities as much as possible [7–9] though analytical approaches are in progress too [10], with the hope that some of which could be violated by a stronger factor.

In this paper, we first introduce a simple but general method for constructing Bell expressions which can be applied to two party experiments each measuring two observables with different outputs. Generalization to more parties and/or more observables is straightforward. Then in a special case of a two-party system with two settings and two possible outcomes on each side, we use these expressions and show that according to local theories there exist *equalities* which are violated by quantum theory by a stronger factor than Bell-type inequalities. Meanwhile we derive an inequality in this case which is violated by a factor of 1.621 tolerating 0.293 fraction of white noise admixture. Although in this case the tolerance is the same as those derived previously in the literature, the range of violation is greater.

2. Derivation of Bell-type expressions

Let us consider a two-party system with no known interaction between its parts. Suppose the left party, say, A performs two possible measurements a with outcomes $i \in \{0, \ldots, m-1\}$ and a' with outcomes $i' \in \{0, \ldots, m'-1\}$. Similarly, the right party, say B, performs two possible measurements b with outcomes $j \in \{0, \ldots, n-1\}$ and b' with outcomes $j' \in \{0, \ldots, n'-1\}$. For simplicity, from now on we use unprimed/primed variables and indices for the first/second measurement for each party whenever applicable and label such a system as $mn \otimes m'n'$. As a consequence of locality, for measurements which are not simultaneous, there exists a probability $q_{aa'bb'}^{ii'jj'}$ defined as the probability that measurements of a result i, a' result i', b result j and b' result j'. The total number of q's, N_Q, is $N_Q = m \times m' \times n \times n'$. As q's are all exclusive and cover all probable events, with the assumption that the probability distribution for each measurement is normalized, we have

$$\sum_{i,i',j,j'} q_{aa'bb'}^{ii'jj'} = 1. \tag{1}$$

These q's are the building blocks of all possible Bell expressions for local theories.

If P_{ab}^{ij} denotes the probability that in a particular experiment A measures a with outcome i and B measures b with outcome j, we can write

$$P_{ab}^{ij} = \sum_{i',j'} q_{aa'bb'}^{ii'jj'}, \qquad \text{and so on} \ldots. \tag{2}$$

There are totally N_P number of P's, where $N_P = (m+m')(n+n')$. Equation (2) in matrix form would become

$$\mathbf{P} = \mathbf{MQ} \tag{3}$$

where $\mathbf{P}$ is an $N_P \times 1$ column matrix, $\mathbf{Q}$ is an $N_Q \times 1$ column matrix and $\mathbf{M}$ is the conversion matrix with dimension $N_P \times N_Q$. These P's are not all independent. The rank of the matrix $\mathbf{M}$, denoted by N_I, is the number of independent P's. For $22 \otimes 22$, $23 \otimes 22$ and $23 \otimes 23$, the rank of the conversion matrix, $\mathbf{M}$, is 9, 12 and 16, respectively. Note that these do not agree with the results obtained in [11] in which the number of independent P's for $22 \otimes 22$ case is predicted to be 8, and for $23 \otimes 22$ case it is 11 (for $m = 2$, $m' = 3$, $n = 2$ and $n' = 2$) or 14 (for $m = 2$, $m' = 2$, $n = 2$ and $n' = 3$) and for $23 \otimes 23$ case it is 20 (for $m = 2$, $m' = 3$, $n = 2$ and $n' = 3$) or 19 (for $m = 3$, $m' = 2$, $n = 2$ and $n' = 3$).

To prove these numerical results analytically we search for constraints which justify the rank of the matrix $\mathbf{M}$. Two groups of constraints are directly derived from equations (1) and (2).

The first is *normalization* of probability distribution in a measurement. That is

$$\sum_{i,j} P_{ab}^{ij} = 1, \tag{4}$$

$$\sum_{i,j'} P_{ab'}^{ij'} = 1, \tag{5}$$

$$\sum_{i',j} P_{a'b}^{i'j} = 1, \tag{6}$$

$$\sum_{i',j'} P^{i'j'}_{a'b'} = 1. \tag{7}$$

However, with the assumption of CH (see [6]) we can write

$$P^{ij}_{ab} P^{i'j'}_{a'b'} = P^{i'j}_{a'b} P^{ij'}_{ab'} = q^{ii'jj'}_{aa'bb'}, \tag{8}$$

and from equation (1) we have

$$\sum_{i,i',j,j'} P^{ij}_{ab} P^{i'j'}_{a'b'} = \sum_{i,i',j,j'} q^{ii'jj'}_{aa'bb'} = 1, \tag{9}$$

$$\sum_{i,i',j,j'} P^{i'j}_{a'b} P^{ij'}_{ab'} = \sum_{i,i',j,j'} q^{ii'jj'}_{aa'bb'} = 1. \tag{10}$$

So, for instance, equations (4), (5) and (9) can be considered as independent and these equations impose only three independent constraints on P's. Note that in this case equation (10) is automatically satisfied and equation (6) can be derived from equations (5) and (10). Similarly, equation (7) can be derived from equations (4) and (9).

The second group of constraints derived from equations (1) and (2) are

$$\sum_{j} P^{ij}_{ab} = \sum_{j'} P^{ij'}_{ab'}, \tag{11}$$

$$\sum_{j} P^{i'j}_{a'b} = \sum_{j'} P^{i'j'}_{a'b'}, \tag{12}$$

$$\sum_{i} P^{ij}_{ab} = \sum_{i'} P^{i'j}_{a'b}, \tag{13}$$

$$\sum_{i} P^{ij'}_{ab'} = \sum_{i'} P^{i'j'}_{a'b'}, \tag{14}$$

which imply *no-signalling*. The total number of constraints from the above equations add up to $m + m' + n + n'$. However in each group of the above constraints, one of them is a linear combination of the other, for example from equations (4) and (5), one can write

$$\sum_{i,j} P^{ij}_{ab} = \sum_{i,j'} P^{ij'}_{ab'}, \tag{15}$$

and from equation (11) we have

$$\sum_{i\neq l,j} P^{ij}_{ab} = \sum_{i\neq l,j'} P^{ij'}_{ab'}. \tag{16}$$

Subtracting equation (16) from equation (15) results in

$$\sum_{j} P^{lj}_{ab} = \sum_{j'} P^{lj'}_{ab'}, \tag{17}$$

which is one of the constraint in equation (11). So, the constraints from no-signalling would be $(m-1) + (m'-1) + (n-1) + (n'-1)$.

The total number of constraints, N_C, given by equations (4)–(14) would become

$$N_C = m + m' + n + n' - 1. \tag{18}$$

So there are only N_I number of independent P's, where

$$\begin{aligned} N_I &= N_P - N_C \\ &= (m+m')(n+n') - (m+m'+n+n'-1), \end{aligned} \tag{19}$$

which agrees with the numerical results mentioned before and is symmetric with respect to interchanging m and m' (and of course n and n'). Clearly, without CH assumption, i.e. equation (8), we could not get this result. We would like to emphasize that this assumption is only used here to prove equation (19). It has nothing to do with the rest of this paper, especially the main results which will be obtained later.

Generally, if $\mathbb{B}$ is a Bell expression for local theories and $-d \leqslant \mathbb{B} \leqslant c$ with non-negative c and d, then $\mathbb{B}$ must satisfy

$$\begin{aligned} \mathbb{B} &= \sum_{s,t,k,l} \lambda_{stkl} P_{st}^{kl} \\ &= \sum_{i,i',j,j'} (\mu_{ii'jj'} - \nu_{ii'jj'}) q_{aa'bb'}^{ii'jj'}, \qquad \mu_{ii'jj'} \neq \nu_{ii'jj'} \end{aligned} \tag{20}$$

where $c(d)$ is the greatest of non-negative real numbers μ's (ν's).

We have solved equation (3) numerically for $22 \otimes 22$, $23 \otimes 22$ and $23 \otimes 23$ cases to find all possible expressions that satisfy equation (20) and the complement of each one, i.e. all pair of expressions whose sum add up to 1. However, we do not discuss it here because the results that we are going to use in the next section can be tested directly and easily.

3. Violation of equalities and inequalities

Using the numerical method mentioned in the previous section, we have found two expressions in $22 \otimes 22$ case which are complement of each other. These are

$$\left|P_{11}^{10} - P_{12}^{11} + P_{21}^{11} + P_{22}^{01}\right| = q_{1212}^{0001} + q_{1212}^{0011} + q_{1212}^{0110} + q_{1212}^{0111} + q_{1212}^{1000} + q_{1212}^{1001} + q_{1212}^{1100} + q_{1212}^{1110}, \tag{21}$$

and

$$\left|-P_{11}^{01} + P_{12}^{00} + P_{21}^{01} + P_{22}^{11}\right| = q_{1212}^{0000} + q_{1212}^{0010} + q_{1212}^{0100} + q_{1212}^{0101} + q_{1212}^{1010} + q_{1212}^{1011} + q_{1212}^{1101} + q_{1212}^{1111}. \tag{22}$$

Note that all q's are non-negative numbers. Adding these two equations and then using equation (1) would give

$$\left|P_{11}^{10} - P_{12}^{11} + P_{21}^{11} + P_{22}^{01}\right| + \left|-P_{11}^{01} + P_{12}^{00} + P_{21}^{01} + P_{22}^{11}\right| = 1. \tag{23}$$

See appendix A for a direct proof of equations (21), (22) and the above equality. (Though the equality (23) can also be derived analytically from equations (4)–(14) easily.) Of course, due to symmetry, there are other equations of this type as well which we do not mention here for brevity.

Now we use the same experiment used by CH [6] to show that *equality* (23) is violated by quantum theory. Consider a two-photon system in the state

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}} \left[\begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\\0\end{pmatrix} \right] \tag{24}$$

where one photon moves to the left in the $+z$ direction and the other moves to the right in $-z$ direction. The projection of $|\Psi_0\rangle$ on directions $\mathbf{u}(\theta) = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} + 0\mathbf{k}$ (for the left photon) and $\mathbf{v}(\phi) = \cos\phi\mathbf{i} + \sin\phi\mathbf{j} + 0\mathbf{k}$ (for the right photon) is

$$|\Psi(\theta,\phi)\rangle = \begin{pmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta) & 0 \\ \cos(\theta)\sin(\theta) & \sin^2(\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \cos^2(\phi) & \cos(\phi)\sin(\phi) & 0 \\ \cos(\phi)\sin(\phi) & \sin^2(\phi) & 0 \\ 0 & 0 & 0 \end{pmatrix} |\Psi_0\rangle.$$

So,

$$|\Psi(\theta,\phi)\rangle = \frac{1}{\sqrt{2}}\cos(\theta-\phi)\begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix} \otimes \begin{pmatrix}\cos\phi\\ \sin\phi\\ 0\end{pmatrix}. \tag{25}$$

If, for example, the left photon is not detected in the $\mathbf{u}(\theta)$ direction then it must be detected in the direction $\mathbf{u}(\theta+\pi/2) = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} + 0\mathbf{k}$ which is perpendicular to $\mathbf{u}(\theta)$. So defining $\gamma = \theta+\pi/2$ and $\eta = \phi+\pi/2$, for directions perpendicular to $\mathbf{u}$ and $\mathbf{v}$, respectively, we obtain

$$|\Psi(\gamma,\eta)\rangle = \frac{1}{\sqrt{2}}\cos(\theta-\phi)\begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix} \otimes \begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix}, \tag{26}$$

$$|\Psi(\theta,\eta)\rangle = \frac{1}{\sqrt{2}}\sin(\phi-\theta)\begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix} \otimes \begin{pmatrix}\cos\phi\\ \sin\phi\\ 0\end{pmatrix}, \tag{27}$$

$$|\Psi(\gamma,\phi)\rangle = \frac{1}{\sqrt{2}}\sin(\theta-\phi)\begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix} \otimes \begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix}. \tag{28}$$

Denoting the photon detection in a particular direction with 1 and the non-detection with 0 the joint probabilities would be

$$\begin{aligned}
P_{11}^{11} &= \tfrac{1}{2}\cos^2(\theta-\phi); & P_{11}^{10} &= \tfrac{1}{2}\sin^2(\phi-\theta);\\
P_{11}^{01} &= \tfrac{1}{2}\sin^2(\theta-\phi); & P_{11}^{00} &= \tfrac{1}{2}\cos^2(\theta-\phi);\\
P_{12}^{11} &= \tfrac{1}{2}\cos^2(\theta-\phi'); & P_{12}^{10} &= \tfrac{1}{2}\sin^2(\phi'-\theta);\\
P_{12}^{01} &= \tfrac{1}{2}\sin^2(\theta-\phi'); & P_{12}^{00} &= \tfrac{1}{2}\cos^2(\theta-\phi');\\
P_{21}^{11} &= \tfrac{1}{2}\cos^2(\theta'-\phi); & P_{21}^{10} &= \tfrac{1}{2}\sin^2(\phi-\theta');\\
P_{21}^{01} &= \tfrac{1}{2}\sin^2(\theta'-\phi); & P_{21}^{00} &= \tfrac{1}{2}\cos^2(\theta'-\phi);\\
P_{22}^{11} &= \tfrac{1}{2}\cos^2(\theta'-\phi'); & P_{22}^{10} &= \tfrac{1}{2}\sin^2(\phi'-\theta');\\
P_{22}^{01} &= \tfrac{1}{2}\sin^2(\theta'-\phi'); & P_{22}^{00} &= \tfrac{1}{2}\cos^2(\theta'-\phi').
\end{aligned}$$

If we represent $\theta-\phi$, $\phi'-\theta$ and $\phi-\theta'$ by x, y and z, respectively, then $\theta'-\phi'$, denoted by w, would be $-(x+y+z)$.

For $x = 247.46^\circ$, $y = 67.49^\circ$ and $z = 157.50^\circ$, the value of the second term on the left-hand side of equation (23), predicted by quantum theory, is $|-1/2\sin^2 x + 1/2\cos^2 y + 1/2\sin^2 z + 1/2\cos^2 w| = 0.207$ and if quantum theory is local, according to equation (23), for the first term we must have

$$\left|P_{11}^{10} - P_{12}^{11} + P_{21}^{11} + P_{22}^{01}\right| = 0.793. \tag{29}$$

However, the value of the left-hand side of the above equality predicted by quantum theory is $|+1/2\sin^2 x - 1/2\cos^2 y + 1/2\cos^2 z + 1/2\sin^2 w| = 1.207$. Clearly, equality (29) is violated by quantum theory by a factor of 1.522 which exceeds that of CH results by 0.108.

In the presence of white noise, the density matrix is

$$\rho = \gamma\rho_{\text{noise}} + (1-\gamma)\rho_{QM} \tag{30}$$

where $\rho_{\text{noise}} = (1/4) \times \mathbb{1}$ and $\rho_{QM} = |\Psi\rangle\langle\Psi|$. Here, γ is the Bell expression's tolerance of white noise, i.e., the maximum fraction of white noise admixture for which a Bell expression stops being violated. The joint probability for the above state is

$$\mathcal{P}_{ab}^{ij} = \frac{\gamma}{4} + (1-\gamma)P_{ab}^{ij}, \tag{31}$$

and it is easily seen that the tolerance of equality (29) is 0.586 fraction of white noise which is twice that of CH inequality. This is due to the fact that the tolerance is very sensitive to upper bound which is 1 in CH inequality and 0.793 in our *equality*.

Also using the method mentioned in section 3 we have obtained three inequalities which correspond to different values of c and d in equation (20) which can be tested directly too. One of them, for $c = 1$ and $d = 0$, is Clauser–Horne inequality, which reads

$$0 \leqslant +P_{11}^{10} + P_{12}^{00} + P_{21}^{01} - P_{22}^{00} \leqslant 1. \tag{32}$$

The other, for $c = 1$ and $d = 1$, is

$$-1 \leqslant +2P_{11}^{10} + P_{12}^{00} - P_{12}^{10} - P_{21}^{00} + P_{21}^{01} - P_{22}^{00} - P_{22}^{11} \leqslant 1. \tag{33}$$

Finally, for $c = 2$ and $d = 1$, we found

$$-1 \leqslant -P_{11}^{00} + P_{11}^{01} + P_{11}^{10} + P_{12}^{00} - P_{12}^{10} + P_{12}^{11} + P_{21}^{01} - 2P_{21}^{11} - P_{22}^{00} + 2P_{22}^{10} \leqslant 2. \tag{34}$$

A direct proof of this inequality is shown in appendix B. In terms of x, y, z and w, the inequalities (32), (33) and (34) would become

$$0 \leqslant \tfrac{1}{2}(\sin^2 x + \cos^2 y + \sin^2 z - \cos^2 w) \leqslant 1, \tag{35}$$

$$-1 \leqslant \sin^2 x + \tfrac{1}{2}\cos 2y - \tfrac{1}{2}\cos 2z - \cos^2 w) \leqslant 1, \tag{36}$$

$$-1 \leqslant \tfrac{1}{2} + \tfrac{3}{2}(\sin^2 x - \sin^2 y - \cos^2 z + \sin^2 w) \leqslant 2, \tag{37}$$

respectively.

The value of inequalities (35), (36) and (37) for $x = -67.50^\circ(-3\pi/8)$, $y = 202.50^\circ(9\pi/8)$ and $z = -67.50^\circ(-3\pi/8)$ are +1.207, +1.414 and +2.621, respectively; but for $x = 22.50^\circ(\pi/8)$, $y = -67.50^\circ(-3\pi/8)$ and $z = 22.50^\circ(\pi/8)$ these are −0.207, −1.414 and −1.621, respectively. While the difference between upper bound and lower bound for Clauser–Horne inequality is 1.414, in agreement with the result obtained in [6], in equation (33) the lower bound itself is violated by a factor of 1.414 and the lower bound of equation (34) is violated by a factor of 1.621 which are greater than that obtained by others for $22 \otimes 22$ case (see, e.g., [5, 6]). Note that the difference between the upper bound and the lower bound in the two latter cases is 2.818 and 4.242, respectively, and that the range of violation for the upper bound for inequalities (32), (33) and (34) is 0.207, 0.414 and 0.621, respectively. These are the maximum values that we could reach and despite the greater range of violation inequalities (32), (33) and (34) tolerate 0.293 of white noise which is the same as CH inequality.

4. Conclusion

In this paper, we have used a numerical method to derive all possible Bell-type inequalities in a two-party experiment namely $mm' \otimes nn'$. The dimension of joint probabilities space obtained by this method is not in agreement with the results obtained previously in the literature.

However, we confirmed our numerical results using an analytical method. This was done by adopting the assumption of CH, i.e. equation (8). And on this basis we showed that *normalization* of joint probabilities in a run of experiment and *no-signalling* in local theories are implied by equation (1) which is statistically trivial. It is an open question that if any assumption other than equation (8) could lead to these correct results.

We showed that besides inequalities there are equalities that can be used to test local theories. In $22 \otimes 22$ case, the equality derived here is violated by a factor of 1.522 tolerating 0.586 fraction of white noise admixture which is twice that of the previous results in the literature. However, as in higher dimensions there are inequalities which are violated by a stronger factor (see [8]), we expect there may exist equalities in higher dimensions as well which are violated by a stronger factor.

Among many possible Bell-type inequalities that we derived, we could only find three types in the form of equations (32), (33) and (34) which are violated with the same settings used in [5, 6]. However, there are some other settings too for which these inequalities are violated and for all of them the inequality in the form of equation (34) is violated with a stronger factor of 1.621 tolerating 0.293 fraction of white noise admixture.

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This paper is dedicated to late Professor Euan James Squires, my PhD supervisor, and the author of 'The Mystery of the Quantum World' to whom I owe much more than words can describe.

Appendix A. Direct proof of equality (23)

For $22 \otimes 22$ case, using the definition of joint probability P in terms of q's, i.e. equation (2), there are 16 possible P's which are as follows:

$$
\begin{aligned}
P_{11}^{00} &= q_{1212}^{0000} + q_{1212}^{0001} + q_{1212}^{0100} + q_{1212}^{0101} \\
P_{11}^{01} &= q_{1212}^{0010} + q_{1212}^{0011} + q_{1212}^{0110} + q_{1212}^{0111} \\
P_{11}^{10} &= q_{1212}^{1000} + q_{1212}^{1001} + q_{1212}^{1100} + q_{1212}^{1101} \\
P_{11}^{11} &= q_{1212}^{1010} + q_{1212}^{1011} + q_{1212}^{1110} + q_{1212}^{1111} \\
P_{12}^{00} &= q_{1212}^{0000} + q_{1212}^{0010} + q_{1212}^{0100} + q_{1212}^{0110} \\
P_{12}^{01} &= q_{1212}^{0001} + q_{1212}^{0011} + q_{1212}^{0101} + q_{1212}^{0111} \\
P_{12}^{10} &= q_{1212}^{1000} + q_{1212}^{1010} + q_{1212}^{1100} + q_{1212}^{1110} \\
P_{12}^{11} &= q_{1212}^{1001} + q_{1212}^{1011} + q_{1212}^{1101} + q_{1212}^{1111} \\
P_{21}^{00} &= q_{1212}^{0000} + q_{1212}^{0001} + q_{1212}^{1000} + q_{1212}^{1001} \\
P_{21}^{01} &= q_{1212}^{0010} + q_{1212}^{0011} + q_{1212}^{1010} + q_{1212}^{1011} \\
P_{21}^{10} &= q_{1212}^{0100} + q_{1212}^{0101} + q_{1212}^{1100} + q_{1212}^{1101} \\
P_{21}^{11} &= q_{1212}^{0110} + q_{1212}^{0111} + q_{1212}^{1110} + q_{1212}^{1111} \\
P_{22}^{00} &= q_{1212}^{0000} + q_{1212}^{0010} + q_{1212}^{1000} + q_{1212}^{1010} \\
P_{22}^{01} &= q_{1212}^{0001} + q_{1212}^{0011} + q_{1212}^{1001} + q_{1212}^{1011} \\
P_{22}^{10} &= q_{1212}^{0100} + q_{1212}^{0110} + q_{1212}^{1100} + q_{1212}^{1110} \\
P_{22}^{11} &= q_{1212}^{0101} + q_{1212}^{0111} + q_{1212}^{1101} + q_{1212}^{1111}.
\end{aligned}
$$

Let us write equation (23) as

$$E = |E_1| + |E_2| \tag{A.1}$$

where E_1 and E_2 are

$$E_1 = P_{11}^{10} - P_{12}^{11} + P_{21}^{11} + P_{22}^{01} \tag{A.2}$$

and

$$E_2 = -P_{11}^{01} + P_{12}^{00} + P_{21}^{01} + P_{22}^{11}. \tag{A.3}$$

Replacing P's in terms of q's we get

$$\begin{aligned} E_1 &= +q_{1212}^{1000} + q_{1212}^{1001} + q_{1212}^{1100} + q_{1212}^{1101} - q_{1212}^{1001} - q_{1212}^{1011} - q_{1212}^{1101} - q_{1212}^{1111} \\ &\quad + q_{1212}^{0110} + q_{1212}^{0111} + q_{1212}^{1110} + q_{1212}^{1111} + q_{1212}^{0001} + q_{1212}^{0011} + q_{1212}^{1001} + q_{1212}^{1011} \\ &= q_{1212}^{0001} + q_{1212}^{0011} + q_{1212}^{0110} + q_{1212}^{0111} + q_{1212}^{1000} + q_{1212}^{1001} + q_{1212}^{1100} + q_{1212}^{1110}, \\ E_2 &= -q_{1212}^{0010} - q_{1212}^{0011} - q_{1212}^{0110} - q_{1212}^{0111} + q_{1212}^{0000} + q_{1212}^{0010} + q_{1212}^{0100} + q_{1212}^{0110} \\ &\quad + q_{1212}^{0010} + q_{1212}^{0011} + q_{1212}^{1010} + q_{1212}^{1011} + q_{1212}^{0101} + q_{1212}^{0111} + q_{1212}^{1101} + q_{1212}^{1111} \\ &= q_{1212}^{0000} + q_{1212}^{0010} + q_{1212}^{0100} + q_{1212}^{0101} + q_{1212}^{1010} + q_{1212}^{1011} + q_{1212}^{1101} + q_{1212}^{1111}. \end{aligned}$$

However as q's are all non-negative numbers, $E_1 = |E_1|$ and $E_2 = |E_2|$. So, for local theories we may write

$$\begin{aligned} E &= |E_1| + |E_2| \\ &= +q_{1212}^{0001} + q_{1212}^{0011} + q_{1212}^{0110} + q_{1212}^{0111} + q_{1212}^{1000} + q_{1212}^{1001} + q_{1212}^{1100} + q_{1212}^{1110} \\ &\quad + q_{1212}^{0000} + q_{1212}^{0010} + q_{1212}^{0100} + q_{1212}^{0101} + q_{1212}^{1010} + q_{1212}^{1011} + q_{1212}^{1101} + q_{1212}^{1111} \\ &= 1, \end{aligned}$$

where the last equality is implied by equation (1) and this justifies equation (23).

Appendix B. Direct proof of inequality (34)

With a similar procedure used in appendix A, for inequality (34) we have

$$\begin{aligned} J &= -P_{11}^{00} + P_{11}^{01} + P_{11}^{10} + P_{12}^{00} - P_{12}^{10} + P_{12}^{11} + P_{21}^{01} - 2P_{21}^{11} - P_{22}^{00} + 2P_{22}^{10} \\ &= -q_{1212}^{0000} - q_{1212}^{0001} - q_{1212}^{0100} - q_{1212}^{0101} + q_{1212}^{0010} + q_{1212}^{0011} + q_{1212}^{0110} + q_{1212}^{0111} + q_{1212}^{1000} + q_{1212}^{1001} \\ &\quad + q_{1212}^{1100} + q_{1212}^{1101} + q_{1212}^{0000} + q_{1212}^{0010} + q_{1212}^{0100} + q_{1212}^{0110} - q_{1212}^{1000} - q_{1212}^{1010} - q_{1212}^{1100} - q_{1212}^{1110} \\ &\quad + q_{1212}^{1001} + q_{1212}^{1011} + q_{1212}^{1101} + q_{1212}^{1111} + q_{1212}^{0010} + q_{1212}^{0011} + q_{1212}^{1010} + q_{1212}^{1011} \\ &\quad - 2\left(q_{1212}^{0110} + q_{1212}^{0111} + q_{1212}^{1110} + q_{1212}^{1111}\right) - q_{1212}^{0000} - q_{1212}^{0010} - q_{1212}^{1000} - q_{1212}^{1010} \\ &\quad + 2\left(q_{1212}^{0100} + q_{1212}^{0110} + q_{1212}^{1100} + q_{1212}^{1110}\right) \\ &= -\left(q_{1212}^{0000} + q_{1212}^{0001} + q_{1212}^{0101} + q_{1212}^{0111} + q_{1212}^{1000} + q_{1212}^{1010} + q_{1212}^{1110} + q_{1212}^{1111}\right) \\ &\quad + 2\left(q_{1212}^{0010} + q_{1212}^{0011} + q_{1212}^{0100} + q_{1212}^{0110} + q_{1212}^{1001} + q_{1212}^{1011} + q_{1212}^{1100} + q_{1212}^{1101}\right). \end{aligned}$$

Since according to equation (1) each of the parentheses on the right-hand side of the above equation is less than 1 and greater than 0, it is easily seen that $-1 \leqslant J \leqslant 2$ which is inequality (34).

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